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PLANE PROBLEM OF CONVECTIVE HEAT TRANSFER IN A NONLINEAR MEDIUM *

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A problem of convective heat transfer in a perfect fluid flowing past a heated body is solved in the thermal boundary layer approximation, for the case when the dependence of the heat conductivity coefficient of the fluid on temperature obeys a power law. The existence is shown of a steady state surface at some distance from the body, the surface separating the perturbed region from the unperturbed region into which thermal perturbations generated by the heated body do not penetrate.

Let us consider a hot cylindrical body the cross section S of which is bounded, in the xy -plane, by the contour $\partial\Omega$. Let this body be placed in a homogeneous flow of a perfect incompressible fluid moving in the direction of the x-axis. The body in the flow generates a perturbation and upsets its homogeneity. We shall assume that the flow past the body is potential, and denote by $\varphi(x, y)$ and $\psi(x, y)$ the velocity potential and the stream function of this flow, both of dimension of length. Remembering that these functions can be determined for a specified contour $\partial\Omega$ using the methods of the functions of complex variable /1/, we shall assume the functions φ and ψ both known and express the components of the fluid velocity in terms of φ and ψ by

$$v_x = v_0 \frac{\partial\varphi}{\partial x} = v_0 \frac{\partial\psi}{\partial y}, \quad v_y = v_0 \frac{\partial\varphi}{\partial y} = -v_0 \frac{\partial\psi}{\partial x} \tag{1}$$

In the general case we assume that ψ is equal to zero on the contour $\partial\Omega$ and the velocity potential φ on $\partial\Omega$ is known and varies from zero at the stagnation point, to some value L at the point of convergence of the flow.

In solving the problem of convective heat transfer in a moving fluid, we shall assume that the temperature of the incoming flow is zero, and the temperature at the cylinder surface is a known function $u=f(P)$, $P \in \partial\Omega$. We require to find the temperature field within the fluid.

The solution of the present problem was considered in /2/ within the framework of the linear theory of heat conduction. In the present paper we study the heat transfer in a nonlinear medium under the conditions when the heat conductivity coefficient k of the fluid varies with temperature according to the power law

$$k = k_0 (u / u_0)^\sigma, \quad k_0, \sigma > 0 \tag{2}$$

where u_0 denotes the characteristic temperature of the surface of the body in the flow. In this case, the problem of determining the steady state temperature field in a moving fluid, reduces to that of solving a nonlinear boundary value problem in the region $\Omega = R^2 \setminus S$. The problem has the form /3/

$$\begin{gathered} \operatorname{div}(u\mathbf{v}) = k_0 (\rho c)^{-1} \operatorname{div}[(u / u_0)^\sigma \operatorname{grad} u] \\ u = f(P), \quad P \in \partial\Omega, \quad u \to 0, \quad x^2 + y^2 \to +\infty \end{gathered} \tag{3}$$

where ρ is density and c is the specific heat capacity of the fluid.

As was shown in /4,5/, the velocity of propagation of the thermal perturbations in the nonlinear medium in question is finite, and the rate of motion of the thermal wave front decreases with increasing distance from the body. Since the rate of propagation of the thermal wave front through a moving nonlinear medium decreases in the directions opposite to the motion of the medium /6/, we can expect that the thermal wave will penetrate the moving medium in these directions only to a finite depth. In the steady state problem (3) this will lead to the appearance of a stationary surface of perturbation Σ (a line in the xy-plane) near the hot body. The surface separates the region penetrated by the thermal perturbations emitted by the body, in which $u > 0$, from the unperturbed region in which $u = 0$ (see Fig.1 in which the perturbed region is shaded).

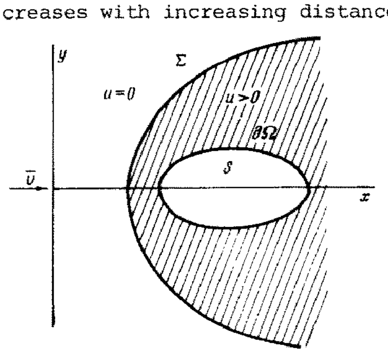

Fig.1

This effect appearing in the heat conduction problem for a nonlinear moving medium is analogous to the effect of formation of a stationary surface of perturbation when a compressible gas flows past a body at a supersonic speed, or to the effect of formation of such a surface when a charged particle

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Let us perform the Boussinesq transformation $(x, y) \to (\varphi, \psi)$. Then, introducing the dimensionless quantities $u' = u / u_0$, $\varphi' = \varphi / L$, $\psi' = \psi / L$ (in what follows we shall omit the primes accompanying the dimensionless quantities), we can pass from problem (3) to the nonlinear problem

$$\frac{\partial u}{\partial \varphi} = \frac{1}{\mathrm{Pe}}\left[\frac{\partial}{\partial \varphi}\left(u^\sigma \frac{\partial u}{\partial \varphi}\right) + \frac{\partial}{\partial \psi}\left(u^\sigma \frac{\partial u}{\partial \psi}\right)\right], \quad \mathrm{Pe} = \frac{v_0 L}{k_0 (\rho c)^{-1}} \tag{4}$$
$$u = F^{\pm}(\varphi), \quad \psi = 0 \pm 0, \quad 0 < \varphi < 1, \quad u \to 0, \quad \varphi^2 + \psi^2 \to +\infty$$

We solve the problem (4) in the thermal boundary layer approximation, assuming that the condition $\mathrm{Pe} \gg 1$ where Pe is the Peclet number, holds. The problem (4) is now reduced to the following problem for the function $u(\varphi, \psi)$ in the region $R_+^2 = \{(\varphi, \psi): \varphi \geqslant 0, \psi \geqslant 0\}$:

$$\frac{\partial u}{\partial \varphi} = \frac{1}{\mathrm{Pe}} \frac{\partial}{\partial \psi}\left(u^\sigma \frac{\partial u}{\partial \psi}\right) \tag{5}$$
$$u = F^+(\varphi), \quad \psi = 0, \quad \varphi > 0, \quad u = 0, \quad \varphi = 0, \quad u \to 0, \quad \psi \to +\infty$$

and to a similar problem in the region $R_-^2 = \{(\varphi, \psi): \varphi \geqslant 0, \psi \leqslant 0\}$.

From the physical point of view, problem (5) corresponds to a problem of a thermal boundary layer near a semi-bounded hot plate placed in a homogeneous flow of a perfect incompressible fluid moving in the direction parallel to the plate surface. Here the dependence of the heat conductivity coefficient on temperature obeys a power law.

Results of /7/ imply that the solution of (5) with $\sigma \neq 0$ for the case of a power relation $F^+(\varphi) = \varphi^\alpha$, can be written in the form of an asymptotic expansion

$$u(\varphi, \psi) = \begin{cases} \varphi^\alpha \left(1 - \dfrac{\psi}{\Lambda \varphi^m}\right)^{1/\sigma} B^{-1} \displaystyle\sum_{i=0}^{\infty} b_i \left(1 - \dfrac{\psi}{\Lambda \varphi^m}\right)^i, & \psi < \Lambda \varphi^m \\ 0, & \psi \geqslant \Lambda \varphi^m \end{cases} \tag{6}$$

$$m = \frac{\alpha\sigma + 1}{2}, \quad B = \sum_{i=0}^{\infty} b_i, \quad \Lambda^2 = [m\sigma B^\sigma \mathrm{Pe}]^{-1}$$

The coefficients $b_i = b_i(\sigma, \alpha)$ are coefficients of expansion of certain function

$$f(\xi) = B^{-1}(1 - \xi)^{1/\sigma} \sum_{i=0}^{\infty} b_i (1 - \xi)^i$$

into a series near the point $\xi = 1$, the function satisfying a nonlinear differential equation and the boundary conditions

$$\alpha f - m\xi \frac{df}{d\xi} = m\sigma B^\sigma \frac{d}{d\xi}\left[f^\sigma \frac{df}{d\xi}\right] \tag{7}$$
$$f(0) = 1, \quad f(1) = 0$$

From (7) we obtain the recurrent relations from which we find the coefficients b_i. In particular, we have /7/

$$b_0 = 1, \quad b_1 = \frac{\alpha\sigma - m}{2m\sigma(\sigma + 1)}, \quad b_2 = -b_1 \frac{1 + 0.5 b_1 (6\sigma^2 + \sigma - 3)}{3(2\sigma + 1)}$$

In the special case $\alpha = \sigma^{-1}$ all $b_i = 0$ for $i \geqslant 1$, and the exact solution of the problem can be written in the simple analytic form

$$u(\varphi, \psi) = \begin{cases} \varphi^{1/\sigma}(1 - \psi/(\lambda\varphi))^{1/\sigma}, & \psi < \lambda\varphi \\ 0, & \psi \geqslant \lambda\varphi \end{cases} \quad (\lambda = (\sigma \mathrm{Pe})^{-1/2})$$

From the form of the solution (6) it follows that a stationary surface (line on the xy- or $\varphi\psi$-planes) exists in a moving nonlinear medium $(\sigma > 0)$, separating the perturbed region near the heated body where $u > 0$, from the unperturbed region where $u = 0$. On the $\varphi\psi$-plane the eqution of this line has the form

$$\psi = \Lambda \varphi^m, \quad m = {}^1/_2 (\alpha\sigma + 1) > 0$$

We note that at the surface of perturbation which represents, in general, a surface of weak discontinuity of the function u, the conditions of continuity of temperature and thermal flux both hold.

Fig.2 depicts, for various values of the power index α corresponding to the different laws governing the temperature change along the surface of the body, with the perturbation lines depicting the quantitative form of the thermal boundary layer. We see that, for a given value of α, the change in the nonlinearity parameter σ affects significantly the spatial structure of the thermal boundary layer.

From the physical point of view, the presence of such a stationary surface of perturbation means that the thermal boundary layer near the hot body is of finite thickness. This is

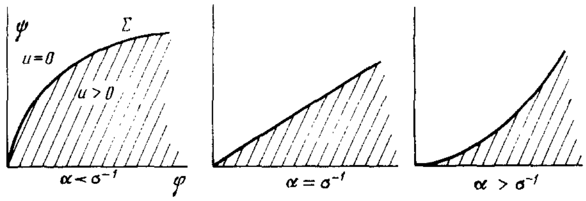

Fig.2

due to the fact that the thermal perturbations propagate through a nonlinear medium with a finite velocity and drift downstream with the moving medium, penentrating into it in the ψ direction only to a finite distance.

The thickness of the thermal boundary layer decreases with increasing Peclet number. On the other hand, when $\sigma \to 0$ then $\Lambda \to +\infty$ and the temperature in a fluid with constant heat conductivity coefficient tends to zero only asymptotically at infinity. This is caused by the fact that the thermal perturbations generated by the plate in such a linear medium, propagate with infinite velocity and penetrate the moving medium infinitely far.

The temperature distribution (6) in the thermal boundary layer obtained above makes possible the estimation of the intensity of the convective heat transfer in the problem in question. We shall restrict ourselves, for simplicity, to the case $\alpha = 0$ when the surface temperature of the body in the flow is constant and equal to u_0.

The local thermal flux q at the plate surface can be found in the case of $\alpha = 0$ from the expression

$$q = -\frac{k_0 u_0}{L\Lambda \sqrt{\varphi}} \left(\frac{df}{d\xi}\right)\Big|_{\xi=0} = \frac{k_0 u_0}{L\Lambda B \sqrt{\varphi}} \sum_{i=0}^{\infty} b_i \left(i + \frac{1}{\sigma}\right) \tag{8}$$

Integrating the heat transfer along the line L on the upper surface of the plate and taking (8) into account, we obtain the following expression for the integral Nusselt number:

$$\mathrm{Nu}(\sigma) = \frac{Q}{k_0 u_0} = \frac{2}{B\Lambda} \sum_{i=0}^{\infty} b_i \left(i + \frac{1}{\sigma}\right) = \mathrm{Nu}(0) \sqrt{\frac{\pi B^{\sigma-2}}{2\sigma}} \sum_{i=0}^{\infty} b_i (1 + i\sigma) \tag{9}$$

Here $\mathrm{Nu}(0) = 2\sqrt{\mathrm{Pe}/\pi}$ is the Nusselt number for the case of a fluid with constant heat conductivity coefficient flowing past a plate.

For $\sigma = 1$, the coefficients b_i have the following values /8/: $b_0 = 1$, $b_1 = -0.25$, $b_2 = 0.014$, $b_3 = 0.0069$, $b_4 = -0.0064$. Therefore, from (9) we find that $\mathrm{Nu}(1)/\mathrm{Nu}(0) = 0.77$. Similar computation for $\sigma = 4$ yields $\mathrm{Nu}(4)/\mathrm{Nu}(0) = 0.56$.

We note that the solution of the problem (5) of the thermal boundary layer, which represented the principal term of the asymptotics of problem (4) with respect to the parameter Pe^{-1}, does not describe the temperature field near the stagnation point of the flow. This must be found from the exact solution of (4). The solution may be obtained e.g. by numerical methods described in /7/. Using the results of /6/, we can establish the dependence of the width of the perturbation region near the stagnation point, on the parameters of the problem, with the relationship given in the form $H \sim L\sigma^{-1}\mathrm{Pe}^{-1}$.

In conclusion we note that problem (5) also describes, in the diffusion boundary layer approximation, the process of mass transfer in a moving medium when the diffusion coefficient is a power function of the concentration.

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